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Geometry of Torsion

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We show that the metric of internal space has an interpretation as a potential for the torsion in a vector bundle (E, M, Λ); the relations among the spin, the local metric of internal space, and the torsion are briefly discussed.

1. INTRODUCTION

Cartan (1922) first investigated gravitation with torsion. He thought that the spin angular momentum is the source of the torsion field, but the torsion field cannot be separated from matter and cannot propagate in the vacuum. Others (e.g., Hammond, 1990; Hoiman *et al.*, 1979) also presented gravitation theories with propagating torsion. Lindstrom (1986) explained the scalar field in the Brans–Dicke theory as the potential of the torsion. Madore (1990) presented a modification of the traditional Kaluza–Klein theory by noncommutative geometry based on a semisimple algebra. He described the union of space-time and an internal space by a principal bundle. Although gravitation with torsion has been investigated extensively, the mechanism of production of the torsion in physics and geometry still is not very clear. In this paper, we investigate this problem in the framework of vector bundles.

2. VECTOR BUNDLE, CONNECTION, CURVATURE, TORSION

We construct a vector bundle (E, M, Λ), where E is the bundle manifold, M is the space-time base manifold, and Λ is the projection $\Lambda: E \to M$. The fiber $V = R^q$ is the q-dimensional vector space; here V is regarded as the internal space. We introduce a local cross section:

$$s: U \subset M \to E \tag{1}$$

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satisfying

$$\Lambda \circ S = \text{id}: U \to U \tag{2}$$

let U be a coordinate locality of M. The local coordinate is x^u ($u = 0, 1, 2, 3$). The q local cross sections S_i ($1 \leq i \leq q$) in the bundle space E on U are chosen as a basis set of the internal space. These q local cross sections are linearly independent. Evidently, at each point $P \in U$, $\{dx^u \otimes s_i,\ u = 0, 1, 2, 3,\ 1 \leq i \leq q\}$ form a basis set of the tensor space $T_p^* \otimes E$.

The connection on the vector bundle is a mapping

$$D: \Gamma(E) \to \Gamma(T^*_{(M)} \otimes E) \tag{3}$$

where $\Gamma(E)$ is the set of sections in the bundle space E on M and satisfies the following conditions:

$$D(s_1 + s_2) = Ds_1 + Ds_2 \tag{4}$$

$$D(as) = da \otimes s + aDs \tag{5}$$

Because Ds_i is a local cross section in the vector bundle $T^*_{(M)} \otimes E$ on U, we have

$$Ds_i = \Gamma^j_{iu}\, dx^u \otimes s_j \tag{6}$$

The connection one-form is defined by

$$W^i_j = \Gamma^i_{ju} dx^u \tag{7}$$

Then (3) becomes

$$Ds_i = W^j_i \otimes s_j \tag{8}$$

For simplicity, we introduce matrix symbols S and W:

$$S = \begin{pmatrix} S_1 \\ \vdots \\ S_q \end{pmatrix}, \qquad W = \begin{pmatrix} W^1_1 & \cdots & W^q_1 \\ \vdots & \vdots\vdots\vdots & \vdots \\ W^1_q & \cdots & W^q_q \end{pmatrix} \tag{9}$$

S denotes the column matrix of the basis of the internal space.

Therefore (6) becomes

$$Ds = w \otimes s \tag{10}$$

We assume $s' = {}^t(s'_1 \cdots s'_q)$ is another basis set of the internal space. We have

$$s' = A.s \tag{11}$$

where

$$A = \begin{pmatrix} a_1^1 & \cdots & a_1^q \\ \vdots & \vdots\vdots\vdots & \vdots \\ a_q^1 & \cdots & a_q^q \end{pmatrix}$$

which satisfies $\det|A| \neq 0$, where a_i^j is a function of the coordinates of the space-time.

Let the matrix of the connection D corresponding to the local cross sections s' be w'. From (5) we have

$$\begin{aligned} Ds' &= dA \otimes s + A.Ds \\ &= (dA.A^{-1} + A.w.A^{-1}) \otimes S' \end{aligned} \tag{12}$$

Therefore

$$w' = dA.A^{-1} + A.w.A^{-1} \tag{13}$$

which is the transformation formula of the connection matrix when the basis of the internal space changes.

We define the curvature matrix corresponding to the connection D as

$$\Omega = dw - w \wedge w \tag{14}$$

By the external differential of (13), we have

$$dw'.A - w' \wedge dA = dA \wedge w + A.dw \tag{15}$$

Substituting (13) into (15), we obtain

$$(dw' - w' \wedge w').A = A.(dw - w \wedge w)$$

We find

$$\Omega' = a.\Omega.A^{-1} \tag{16}$$

which is the transformation formula of the curvature matrix when the basis of the internal space changes.

In order to define torsion, we first introduce a vierbein field e^i_μ (the Latin letters $i, j, k, \ldots$ denote the indices of the coordinates of the internal space, and the Greek letters $\mu, \nu, \ldots$ denote the indices of the space-time). The vierbein field one-form can be written as

$$e^i = e^i_\mu dx^\mu \tag{17}$$

The vierbein fields satisfy the following relations:

$$\begin{aligned} e^i_\mu e^\mu_j &= \delta^i_j, \qquad & e^i_\mu e^\nu_i &= \delta^\nu_\mu \\ g_{\mu\nu} e^\mu_i e^\nu_j &= g_{ij}, \qquad & g_{ij} e^i_\mu e^j_\nu &= g_{\mu\nu} \end{aligned} \tag{18}$$

where $g_{\mu\nu}$ is the metric of the space-time and g_{ij} is the metric of the internal space.

The torsion defined by

$$Q^i = \nabla e^i = de^i + W^i_j \wedge e^j \qquad (19)$$
$$= \frac{1}{2} Q^i_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

where $Q^i_{\mu\nu} = \nabla_\mu e^i_\nu - \nabla_\nu e^i_\mu$ is the torsion tensor and $\nabla_\mu e^i_\nu = \partial_\mu e^i_\nu + \Gamma^i_{\mu j} e^j_\nu$. The torsion two-form of the space-time is expressed as

$$Q^\rho = Q^i e^\rho_i \qquad (20)$$

3. DISCUSSION AND CONCLUSION

We assume that the metric of the internal space corresponding to the vierbein field e^i_μ is wholly flat, that is, $g_{ij} = \text{diag}(1, \ldots, 1)$, and the torsion of the space-time with respect to the e^i is zero. We have

$$Q^i = de^i + w^i_j \wedge e^j = 0 \qquad (21)$$
$$Q^\rho = Q^i e^\rho_i = 0$$

Now we introduce a new vierbein field E^i_μ,

$$E^i_\mu = A^i_j e^j_\mu \qquad (22)$$

Its reciprocal E^μ_i is

$$E^\mu_i = A^{-1j}_i e^\mu_j \qquad (23)$$

The E^i_μ and its reciprocal E^μ_i also satisfy equation (18).

The metric of the internal space corresponding to the new vierbein field is

$$g'_{ij} = A^{-1k}_i A^{-1l}_j g_{kl} \qquad (24)$$

and its determinant is

$$\det|g'_{ij}| = \det|A^{-1}|^2 \qquad (25a)$$

We let $\det|A^{-1}|^2 = 1/\phi^2$. Therefore we have

$$\det|g'_{ij}| = \frac{1}{\phi^2} \qquad (25b)$$

From (24), we know that the metric of the internal space is of non-Minkowskian form and is a function of the space-time coordinates, that is, the metric of the internal space has been "localized." The connection corresponding to the new vierbein field becomes

$$w_j'^i = dA_k(A^{-1})_j^k + A_k^i w_l^k (A^{-1})_j^l \tag{26}$$

The torsion two-form corresponding to the new vierbein field is

$$\begin{aligned} Q'^i &= \nabla E^i = \nabla(A_j^i e^j) \\ &= dA_j^i \wedge e^j + A_j^i de^j + w_j'^i \wedge A_k^j e^k \end{aligned} \tag{27}$$

Substituting (21), (26) into (27), we get

$$Q'^i = 2dA_j^i \wedge e^j \tag{28}$$

Thus the torsion two-form of the space-time is

$$\begin{aligned} Q'^\rho &= Q'^i E_i^\rho \\ &= 2dA_j^i \wedge e^j e_k^\rho (A^{-1})_i^k \\ &= 2\left[\frac{1}{2}\left(\frac{\partial_\mu \phi}{\phi}\,\delta_\nu^\rho - \frac{\partial_\nu \phi}{\phi}\,\delta_\mu^\rho\right)\right] dx^\mu \wedge dx^\nu \end{aligned} \tag{29}$$

Therefore the torsion tensor of the space-time is

$$Q'^\rho_{\mu\nu} = 2\left(\frac{\partial_\mu \phi}{\phi}\,\delta_\nu^\rho - \frac{\partial_\nu \phi}{\phi}\,\delta_\mu^\rho\right) \tag{30}$$

Equation (30) is the equation of the torsion field. From equation (30), we learn that the metric of the internal space may be understood as a potential for the space-time torsion.

We assume

$$2\left(\frac{\partial_\mu \phi}{\phi}\,\delta_\nu^\rho - \frac{\partial_\nu \phi}{\phi}\,\delta_\mu^\rho\right) = k\tau^\rho_{\mu\nu} \tag{31}$$

Therefore from (30) we get

$$Q'^\rho_{\mu\nu} = k\tau^\rho_{\mu\nu}$$

We explain $\tau^\rho_{\mu\nu}$ as the spin angular momentum tensor of the gauge field. Equation (31) is the same as the Einstein–Cartan equation.

It is interesting that the torsion of space-time is produced by the local metric of the internal space. This seems more profound in its geometry and physics than the concept that the spin of matter produces the torsion of the space-time in Cartan theory. We come to the following conclusion: Both kinds of spin of matter and gauge field "localize" the metric of the internal space in geometry, and the local metric of the internal space produces the torsion of the space-time.

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